Dimostrazione Master Theorem

- N potenza esatta di b (ipotesi)

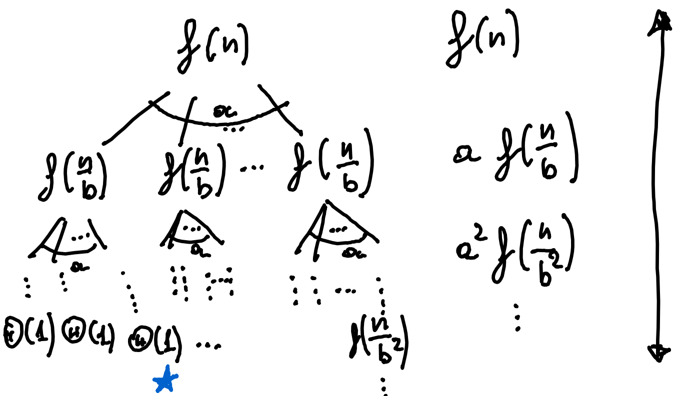
$$T(n) = \begin{cases} \Theta(1)\,\star & n = 1 \\ a\,T\left(\frac{n}{b}\right) + f(n) & n = b^i \end{cases}$$

Numero di sottoproblemi

Dimensione del sottoproblema

costo unione + divisione (semplificato)

Albero di ricorsione



$h = \log_b n + 1$

Σ

$$\Theta\left(a^{\log_b n}\right) = \Theta\left(n^{\log_b a}\right)$$

dato che le foglie sono $n^{\log_b a}$
ed ogni foglia costa 1, il costo è $n^{\log_b a}$

$$T(n) = \underbrace{\Theta\left(n^{\log_b a}\right)}_{\text{costo delle foglie}} + \underbrace{\sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right)}_{\text{il costo effettivo}}$$

$$g(n) = \sum_{j=0}^{\log_b n - 1} a^j \left(\frac{n}{b^j}\right)$$

1. $f(n) = O\left(n^{\log_b a - \varepsilon}\right)$

$$f\left(\frac{n}{b^j}\right) = O\left(\left(\frac{n}{b^j}\right)^{\log_b a - \varepsilon}\right)$$

sostituito nella sommatoria

$$g(n) = O\left(\sum_{j=0}^{\log_b n - 1} a j \left(\frac{n}{b^j}\right)^{\log_b a - \varepsilon}\right)$$

Da qui si formano le tre strategie (i 3 casi):

1. $f(n) = O\left(n^{\log_b a - \varepsilon}\right)$. $\varepsilon > 0$
   $g(n) = O\left(n^{\log_b a}\right)$

2. $f(n) = O\left(n^{\log_b a}\right.$
   $g(n) = \Theta\left(n^{\log_b a} \log n\right)$

3. $a f\left(\frac{n}{b}\right) \leq c f(n)$. $c < 1 \land n > n_0$
   $g(n) = \Theta(f(n))$

e da qui facciamo uso di una serie di trasformazioni algebriche

Sviluppiamo la sommatoria

$$\sum_{j=0}^{\log_b n - 1} a j \left(\frac{n}{b^j}\right)^{\log_b a - \varepsilon} =$$

$$= n^{\log_b a - \varepsilon} \sum_{j=0}^{\log_b n - 1} a j \left(\frac{1}{b^j}\right)^{\log_b a - \varepsilon} =$$

$$= n^{\log_b a - \varepsilon} \sum_{j=0}^{\log_b n - 1} a j \left(\frac{b^{\varepsilon}}{\underbrace{b^{\log_b a}}_{a}}\right)^j =$$

$$= n^{\log_b a - \varepsilon} \sum_{j=0}^{\log_b n - 1} \cancel{a j} \left(\frac{b^{\varepsilon}}{\cancel{a}}\right)^j$$

$$= n^{\log_b a - \varepsilon} \sum_{j=0}^{\log_b n - 1} \left(b^{\varepsilon}\right)^j$$

Serie geometrica!

$$\left(\frac{\overbrace{1^{\log \dots}}^{=1}}{b^{j \log_b a - \varepsilon}}\right)^{\dots} = \left(\frac{1}{b^{\log_b a - \varepsilon}}\right)^j = \left(\frac{1}{b^{\log_b a} - b^{\varepsilon}}\right)^j = \left(\frac{b^{\varepsilon}}{b^{\log_b a}}\right)^j$$

$$= n^{\log_b a - \varepsilon} \sum_{j=0}^{\log_b n - 1} b^{\varepsilon j}$$

$$= n^{\log_b a - \varepsilon} \cdot \frac{\left(b^{\varepsilon}\right)^{\log_b(n)+1} - 1}{b^{\varepsilon} - 1}$$

Serie geometrica!

$$\boxed{\sum_{k=0}^{n} x^{k} = \frac{x^{n+1} - 1}{x - 1}}$$

$$= n^{\log_b a - \varepsilon} \cdot \frac{b^{\varepsilon} \cdot \cancel{b^{\log_b}} n^{\varepsilon} - 1}{b^{\varepsilon} - 1}$$

$$= n^{\log_b a - \varepsilon} \cdot \frac{b^{\varepsilon} \cdot n^{\varepsilon} - 1}{b^{\varepsilon} - 1}$$

$$= n^{\log_b a - \varepsilon} \cdot \frac{b^{\varepsilon} n^{\varepsilon} - 1}{b^{\varepsilon} - 1} \leq n^{\log_b a - \varepsilon} \frac{b^{\varepsilon} n^{\varepsilon}}{b^{\varepsilon} - 1} =$$

$$= n^{\log_b a} \cdot \frac{1}{\cancel{n^{\varepsilon}}} \cdot \frac{b^{\varepsilon} \cancel{n^{\varepsilon}}}{b^{\varepsilon} - 1}$$

$$= n^{\log_b a} \cdot O(1)$$ $b^{\varepsilon} \cong const = -1$

$$= n^{\log_b a - \varepsilon} \; O(n^{\varepsilon})$$

$$= \frac{n^{\log_b a}}{n^{\varepsilon}} \; O(n^{\varepsilon}) = O\left(n^{\log_b a}\right)$$